Today <u>Dual Side</u>

$G = GL(r)$ always

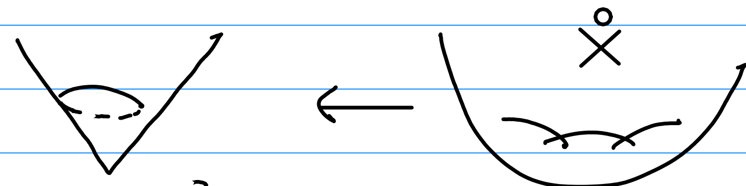
$\Gamma \subset SL_2(\mathbb{C})$ finite subgrp $(\Gamma \neq 1)$

$\longleftrightarrow$ Dynkin diagram of type ADE

$\longleftrightarrow$ Lie algebra $\mathfrak{g}$

Two ways

a) consider



$\mathbb{C}^2/\Gamma$ $\qquad$ $\mathring{X}$ resolution of singularities

: $\mathbb{P}^1 \longleftrightarrow \circ$

$\longleftrightarrow \circ\!-\!\circ$ $\qquad$ Get Dynkin diagram

Let us define $H^2(\mathring{X};\mathbb{Z}) \cong$ weight lattice Λ for $\mathfrak{g}$

$$\begin{array}{ccc} \Psi & & \Psi \\ [C_i] & \longleftrightarrow & \alpha_i \end{array}$$

$-$(intersection matrix) $\longleftrightarrow$ Cartan matrix

b) McKay correspondence

$\{\rho_i\}$: irreducible representations of Γ

(ρ_0 = trivial)

Q : natural 2-dim'l rep. of Γ coming from $\Gamma \subset SL_2$

Define (c_{ij}) by $\rho_i \otimes (\Lambda^0 Q - \Lambda^1 Q + \Lambda^2 Q) = \bigoplus_j c_{ij} \rho_j$

$\Rightarrow$ (c_{ij}) : affine Cartan matrix

Let us identify $R(\Gamma) \cong \hat{\Lambda}$

$\psi_{\rho_i} \longleftrightarrow \Lambda_i$ (fundamental wt)

$\Lambda^* Q \otimes \rho_i \longleftrightarrow \alpha_i$ $\quad \langle \alpha_i, h_j \rangle = c_{ij}$ $\therefore$ $\alpha_i = \sum c_{ij} \Lambda_j$

$0 = \Lambda^* Q \otimes$ regular rep $\leftrightarrow$ $\delta = 0$ in Λ

NB. ① $\dim \rho$ = level ($\because$ $\langle c, \Lambda_i \rangle = a_i$) $\quad c = a_0\alpha_0 + \cdots + a_n\alpha_n$

② $R(\Gamma)_+$: actual representations of Γ $\leftrightarrow$ dominant weight $\hat{\Lambda}_+$

This is different from BF's parametrization for $G = GL(r)$, $\Gamma = \mathbb{Z}_\ell$

but related via the level-rank duality.

Want to construct various partial resolutions of

① $\mathcal{U}^{\lambda}_{\mu}$ = Uhlenbeck space for $\mathbb{C}^2/\Gamma$

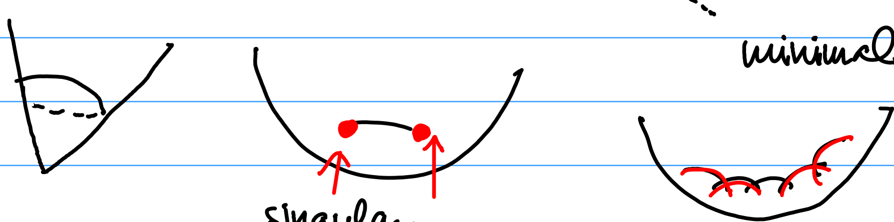
$= \overline{\mathrm{Bun}^{\lambda}_{\mu}}$ in $\mathcal{U}^{k}_{\bar{\mu}}(\mathbb{C}^2)^{\Gamma}$ $\qquad \frac{k}{\ell} = \langle \lambda - \mu, d \rangle$

Choose a subdiagram. e.g. $\underbrace{\bullet - \bullet}_{\ell_1 - 1} - \circ - \underbrace{\bullet - \bullet}_{\ell_2 - 1}$ $\qquad \ell = \ell_1 + \ell_2$

$$\mathbb{C}^2/\Gamma \longleftarrow \exists\, \dot{X} \xleftarrow{\text{partial resolution}} \mathring{X}$$

$\dot{\mathfrak{g}} = \mathfrak{g}_{\dot{I}}$

minimal resolution



singular pt of type $A_{\ell_1 - 1}$, $A_{\ell_2 - 1}$

Dynkin diagram

② $\dot{\mathcal{U}}^{\lambda}_{\mu}$ = Uhlenbeck space for $\dot{X}$

②' $\mathring{\mathcal{U}}^{\lambda}_{\mu}$ = Uhlenbeck space for $\mathring{X}$

③ $\mathring{\mathcal{M}}^{\lambda}_{\mu}$ = Gieseker space for $\mathring{X}$

framed moduli space of torsion-free sheaves on $\mathring{X}$

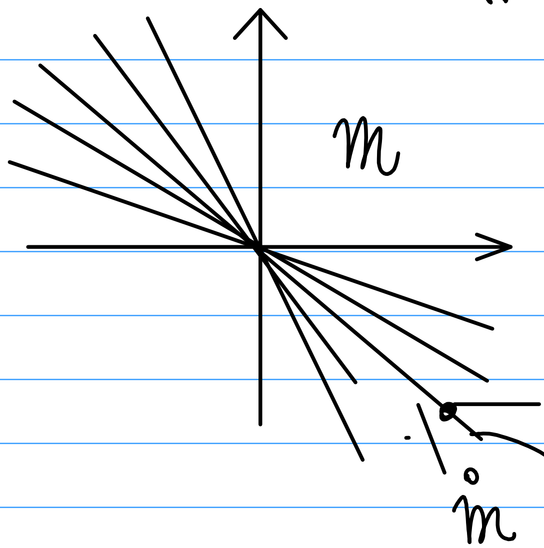
④ $\mathcal{M}^{\lambda}_{\mu}$ = (Gieseker space for $\mathbb{C}^2$)$^{\Gamma}$

Remark

moduli space can be defined for any choice of "stability parameter" $\in \mathfrak{h}_{\mathbb{R}}$

depending on the face defined by the affine root system

$\widehat{sl}_2$



$\mathring{\mathfrak{U}}$ in on the root hyperplane in $\delta = 0$

level 0 hyperplane $\delta = 0$

$\bar{\mu}$: representation $\Gamma \to GL(r) \longleftrightarrow$ dominant weight
this is common in ②, ②$'$, ③, ④

But λ's are different:

③ or ②$'$ $\lambda = \left(\begin{array}{l} c_1 \in H^2(\overset{\circ}{X}; \mathbb{Z}) = \Lambda : \text{weight lattice for } \mathfrak{g} \\ \text{together with } k \end{array} \right.$

$\Rightarrow$ affine weight Λ_{aff}, but not necessary dominant

④ $\lambda = \left(\begin{array}{l} \text{representation of "fiber" at } 0 \\ k \end{array} \right.$

Γ-fixed pt in $\mathbb{C}^2$

Since $\mathcal{M}$ parametrises sheaves, the representation is only virtual.

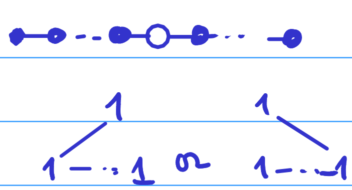
$\therefore \quad \lambda \in R(\Gamma) \times H^4 = \Lambda_{\mathrm{aff}}$

Rem. Since we are fixing r (= rk of sheaves), level of the affine weights are already chosen.
So no distinction between Λ and $\hat{\Lambda}$

② $\lambda = \left\{ \begin{array}{l} c_1 \in H^2(\overset{\bullet}{X}; \mathbb{Z}) = \mathbb{Z}^{I^\circ} \\ + \text{ representation } \quad \mathbb{Z}_{\ell_1} \to G, \ \mathbb{Z}_{\ell_2} \to G \\ \quad \text{at singular point} \quad \uparrow \Lambda(\hat{\mathfrak{sl}}_{\ell_1 - 1})_+ \quad \uparrow \Lambda(\hat{\mathfrak{sl}}_{\ell_2 - 1})_+ \\ k \end{array} \right.$

$\longleftrightarrow I^\bullet$-dominant weight for Λ_{aff} $\rightsquigarrow$ dominant wts of $\mathring{\mathfrak{g}}_{\mathrm{aff}}$ (common central ext.)

i.e. $\left\{ \begin{array}{l} \langle \lambda, h_i \rangle \geq 0 \quad \text{for } i \in I^\bullet \\ \langle \lambda, c - h_{\theta} \rangle \geq 0 \end{array} \right.$



$\uparrow$ highest coroot of the component $\mathcal{C}$ of $I^\bullet$

$$
\begin{tikzcd}
\mathring{\mathcal{M}}^{\lambda}_{\mu} \arrow[r] \arrow[dr, "\pi"'] & \mathring{\mathcal{U}}^{\lambda}_{\mu} \arrow[d] & & \\
& \overset{\bullet}{\mathcal{U}}^{\lambda}_{\mu} \arrow[dr, "\pi'"'] & & \mathcal{M}^{\lambda}_{\mu} \arrow[dl, "\sigma"] \\
& & \mathcal{U}^{\lambda}_{\mu} &
\end{tikzcd}
$$

If λ does not satisfy the dominance condition, one need to replace λ for the target by dominant one in the appropriate "Weyl group" orbit.

<u>Prop.</u> (special feature for $G = GL(r)$)

$\mathcal{M}^{\lambda}_{\mu}$, $\mathring{\mathcal{M}}^{\lambda}_{\mu}$: smooth (diffeomorphic to each other)

But $\mathcal{U}^{\lambda}_{\mu}$, $\cdots$ never smooth except $= \{pt\}$

$$\left(\mathcal{M}^{\mu}_{\mu} = \mathcal{U}^{\mu}_{\mu} = \{pt\} \right)$$

$\uparrow$ trivial connection

Fix μ and move all λ.

$\mathcal{U}^{\lambda}_{\mu} \subset \mathcal{U}^{\lambda'}_{\mu}$ if $\lambda \geq \lambda'$

closed emb.

$\lambda = w - v$

$\lambda' = w - v'$

$\therefore \lambda \geq \lambda' \Leftrightarrow v' \geq v$

$$\left(\begin{array}{l} \lambda - \lambda' \in R(\Gamma)_{+} \times \mathbb{Z} \\[1em] \mathrm{Bun}^{\lambda}_{\mu}(\mathbb{C}^2/\Gamma) \times \mathrm{Bun}^{\bar{\lambda}', k}_{\bar{\lambda}}(\mathbb{C}^2/\Gamma) \dashrightarrow \mathrm{Bun}^{\lambda'}_{\mu}(\mathbb{C}^2/\Gamma) \\ \longrightarrow \mathcal{U}^{\lambda'}_{\mu} \end{array} \right.$$

Taubes

$$\mathcal{U}_{\mu} := \varinjlim \mathcal{U}^{\lambda}_{\mu}$$

$$\begin{array}{ccc} \mathfrak{M}_\mu^\lambda & & \mathfrak{M}_\mu^{\lambda'} \\ & \searrow \quad \swarrow & \\ & \mathfrak{U}_\mu & \end{array}$$

$Z_\mu^{\lambda,\lambda'}$: fiber product $\subset \mathfrak{M}_\mu^\lambda \times \mathfrak{M}_\mu^{\lambda'}$ half-dim.

$\bigoplus_{\lambda,\lambda'} H_{top}(Z_\mu^{\lambda,\lambda'})$: algebra without 1

$1 = $ "$\prod_\lambda \Delta_{\mathfrak{M}_\mu^\lambda}$"

$\Bigg($ $M_1, M_2 \to Y$ $\qquad Z_{12} = M_1 \times_Y M_2$

M_3

$$H_*(Z_{12}) \otimes H_*(Z_{23}) \longrightarrow H_*(Z_{13})$$

$$c_{12} \otimes c_{23} \mapsto p_{13*}(p_{12}^*(c_{12}) \cap p_{23}^*(c_{23}))$$

<u>Th</u> (N: 1998)

1) $\exists\ \tilde{U}((\mathfrak{g}_\Gamma)_{aff}) \longrightarrow \bigoplus_{\lambda,\lambda'} H_{top}(Z_\mu^{\lambda,\lambda'})$ algebra hom.

(modified env. alg.) $\qquad$ (not surjective) injective

2) $\bigoplus_\lambda H_{top}(\mathfrak{L}_\mu^\lambda) \cong V(\mu)$ as a representation of $(\mathfrak{g}_\Gamma)_{aff}$.

$\mathfrak{L}_\mu^\lambda$ $=$ inv. image of $\mathfrak{U}_\mu^\mu = pt$ $\qquad$ s.t. $1 \in H_{top}(\mathfrak{L}_\lambda^\lambda) = H_{top}(pt)$ is the highest wt vector

inv. image of $\mathfrak{U}_\mu^\mu = pt$ $=$ $\mathfrak{M}_\mu^\lambda \times_{\mathfrak{U}_\mu} \mathfrak{M}_\mu^\mu$

$\mathfrak{M}_\mu^\mu$ $\uparrow$ pt

About the construction

$\mathfrak{g}_{\mathrm{aff}}$, as a KM Lie algebra, has Chevalley type generators $\langle e_i, f_i, h_i, d \rangle$

$i = 0, 1, \ldots, n$

$$e_0 = x_{-\theta} \otimes t \qquad f_0 = x_\theta \otimes t^{-1}$$

θ: highest root

root vectors in $\mathfrak{g}_{\mathrm{aff}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$

(This is a nontrivial result, checked by a direct calculation ......

$$e_i \longmapsto \sum_\lambda [\mathcal{P}_i^\lambda] \qquad \mathcal{P}_i^\lambda \subset \mathcal{Z}_\mu^{\lambda, \lambda+\alpha_i}$$

irreducible component

(E, E') $\quad E \subset E'$

$E'/E \cong \mathcal{O}_0 \otimes \rho_i$

$0 \in \mathbb{C}^2$

$$f_i \longmapsto \sum \pm [{}^\omega\mathcal{P}_i^\lambda] \qquad {}^\omega\mathcal{P}_i^\lambda \subset \mathcal{Z}_\mu^{\lambda+\alpha_i, \lambda}$$

$$h_i \longmapsto \sum \lambda [\Delta^\lambda] \qquad \Delta^\lambda \subset \mathfrak{M}_\mu^\lambda \times \mathfrak{M}_\mu^\lambda$$

<u>Remark</u>. $\bigoplus_{\lambda, \lambda'} H_{\mathrm{top}}(\overset{\circ}{\mathcal{Z}}{}_\mu^{\lambda, \lambda'})$ is isomorphic to $\bigoplus_{\lambda, \lambda'} H_{\mathrm{top}}(\mathcal{Z}_\mu^{\lambda, \lambda'})$ as we will see later, but I do not know how to define $U(\mathfrak{g}_{\mathrm{aff}}) \longrightarrow$ directly.

Sheaf theoretic analysis (Borho-MacPherson, Ginzburg)

$\pi : M \longrightarrow X$ semismall

$$\pi_*(\mathbb{C}_M[\dim M]) = \bigoplus_{\alpha,\rho} IC(X_\alpha, \rho) \otimes H_{top}(\pi^{-1}(x_\alpha))^\rho$$

$Z := M \times_X M$ fiber product

Th. $H_{top}(Z) \cong \mathrm{End}_{Perv(X)}(\pi_*(\mathbb{C}_M[\ ]))$

$$= \bigoplus_{\alpha,\rho} \mathrm{End}_{\mathbb{C}}(H_{top}(\pi^{-1}(x_\alpha))^\rho) \quad : \text{semisimple}$$

$H_{top}(\pi^{-1}(x_\alpha))^\rho$: irreducible rep. of $H_{top}(Z)$

◎ branching

$M \xrightarrow{\pi} Y \xrightarrow{\pi'} X \qquad Z_Y := M \times_Y M \subset Z_X := M \times_X M$

$\therefore\ H_{top}(Z_Y) \subset H_{top}(Z_X)$

$$\pi_* \mathbb{C}_M[\ ] = \bigoplus IC(Y_\nu, \psi) \otimes H_{top}(\pi^{-1}(y_\nu))^\psi$$

$$(\pi' \circ \pi)_* \mathbb{C}_M[\ ] = \bigoplus IC(X_\alpha, \rho) \otimes H_{top}((\pi' \circ \pi)^{-1}(x_\alpha))^\rho$$

$$\parallel$$

$$\pi'_*(\pi_* \mathbb{C}_M[\ ]) = \bigoplus \underbrace{\pi'_*(IC(Y_\nu, \psi))}_{= \bigoplus IC(X_\alpha, \rho)^{\oplus m^{\nu,\psi}_{\alpha,\rho}}} \otimes H_{top}(\pi^{-1}(y_\nu))^\psi$$

$$\therefore\ H_{top}(\pi^{-1}(y_\nu))^\psi \otimes m^{\nu,\psi}_{\alpha,\rho} = H_{top}((\pi' \circ \pi^{-1})(x_\alpha))^\rho$$

$$\therefore\ \boxed{[H_{top}(\pi^{-1}(y_\nu))^\psi : \mathrm{Res}\, H_{top}((\pi' \circ \pi^{-1})(x_\alpha))^\rho] = m^{\nu,\psi}_{\alpha,\rho}}$$

<u>Remark</u> $\quad \mathring{\mathfrak{M}}^\lambda_\mu \xrightarrow{\pi'\circ\pi} \mathfrak{U}^\lambda_\mu \xleftarrow{\sigma} \mathfrak{M}^\lambda_\mu$

$$(\pi'\circ\pi)_*(\mathbb{C}_{\mathring{\mathfrak{M}}^\lambda_\mu}[\ ]) \cong \sigma_*(\mathbb{C}_{\mathfrak{M}^\lambda_\mu}[\ ])$$

$\uparrow$

$\exists$ canonical isomorphism

(as 1 param deformation s.t. all the fibers except 0 are isomorphic & small)

$\therefore \quad H_{top}(\mathring{\mathfrak{L}}^\lambda_\mu) \cong H_{top}(\mathfrak{L}^\lambda_\mu)$

$$\therefore \quad \boxed{\bigoplus_{\lambda,\lambda'} H_{top}(\mathring{\mathfrak{M}}^\lambda_\mu \times_{\mathfrak{U}_\mu} \mathring{\mathfrak{M}}^{\lambda'}_\mu) \cong \bigoplus_{\lambda,\lambda'} H_{top}(Z^{\lambda,\lambda'}_\mu)}$$

But the varieties $\mathring{\mathfrak{L}}^\lambda_\mu$, $\mathfrak{L}^\lambda_\mu$ look different

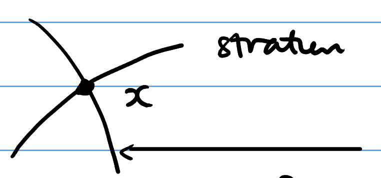
$\mathring{\mathfrak{L}}^\lambda_\mu \subset \mathring{\mathfrak{M}}^\lambda_\mu$, $\quad \mathfrak{L}^\lambda_\mu \subset \mathfrak{M}^\lambda_\mu$

$\mathring{\mathfrak{L}}^\lambda_\mu$ looks more closely related to the description $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c\oplus\mathbb{C}d$, as one can define the Heisenberg alg. operators

(cf. special case
Nagao : Frenkel–Kac construction

We also need to know $\mathring{\mathfrak{M}}^\lambda_\mu \times_{\mathring{\mathfrak{U}}_\mu} \mathring{\mathfrak{M}}^{\lambda'}_\mu$, but

$\forall x \in \mathring{\mathfrak{U}}_\mu \quad \pi : \mathring{\mathfrak{M}}^\lambda_\mu \to \mathring{\mathfrak{U}}_\mu$ is locally isom. to étale



$\mathring{\mathfrak{M}}^{\lambda'}_{\mu'} \to \mathfrak{U}_{\mu'}$ for the affine quiver associated with $\dot{I}$.

$\rightsquigarrow \dot{\mathfrak{g}}_{aff}$

In our situation one can show no non trivial local system appear.

$$\mathcal{U}_\mu = \bigcup_{\lambda:\text{dominant}} \mathcal{U}_\mu^\lambda \times S_\phi^{|\phi|}(\mathbb{C}^2/\Gamma \setminus 0) \qquad \phi: \text{partition}$$

$$\mathring{\mathcal{U}}_\mu = \bigcup_{\lambda:\mathring{I}\text{-dominant}} \mathring{\mathcal{U}}_\mu^\lambda \times S_\phi^{|\phi|}(\mathring{X} \setminus \text{singular pt})$$

$\lambda:\mathring{I}$-dominant $\longrightarrow$ dominant wt of $\mathring{\mathfrak{g}}_{aff}$

$$\pi_*(IC(\mathring{\mathcal{U}}_\mu^\lambda)) = \bigoplus IC(\mathcal{U}_\mu^{\lambda'})^{\oplus m_\lambda^{\lambda'}}$$

Then $V(\lambda')\big\downarrow = \bigoplus V(\lambda)^{\oplus m_\lambda^{\lambda'}}$

Level-Rank duality I. Frenkel, K. Hasegawa

X, Y : vector spaces of $\dim = l, r$

$$\mathcal{L}(X\otimes Y) = X \otimes Y \otimes \mathbb{C}t^{1/2}[t,t^{-1}]$$

$F = \Lambda^{\infty/2}\mathcal{L}(X\otimes Y)$: Fock space

$\circlearrowleft$ Clifford algebra

Get comm. action of $\widehat{sl}(X)_{\text{level } r}$ $\qquad \widehat{sl}(Y)_{\text{level } l}$ $\qquad \hat{a}$: Heis.

Th. (Hasegawa)

$$F \cong \bigoplus_{\lambda \in Y_\ell^r} V^{\widehat{sl}(x)}(\bar{\lambda}) \otimes V^{\widehat{sl}(r)}(\overline{{}^t\lambda}) \otimes H_{|x|}$$